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COMBINED APPROACH TO ANALYZING THE BUCKLING OF IDEAL CYLINDRICAL SHELLS AT GIVEN PERTURBATIONS

N. S. Astapov and V. M. Kornev

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The inevitable initial irregularities arising in the construction of structures or the perturbations in loading cause a premature loss of stability (compared with Euler loading) of elastic constructions and are the main reason for the scatter in experimental data. Research in this field has been carried out by many authors (see, e.g., [1, 2]).

In the present paper a combined (numerical–analytical) approach to the solution of nonlinear boundary problems is expounded, as applied, by way of example, to the study of the buckling of closed ideal cylindrical shells under transverse or hydrostatic loading with perturbations taken into account. Numerical analysis of the buckling process under transverse and hydrostatic loading shows that the process of successive loading of a shell is accompanied by the distortion of the initial section of the critical load spectrum and by the reconstruction of buckling modes.

1. Classification of the Problems and Description of the Combined Approach. Practical recommendations [1] on calculating the stability of bars, plates, and shells are most frequently oriented to the Euler classical critical load with a certain correcting multiplier to be chosen within the range from 1 to 1/10 depending both on the type of problem and on the possible initial constructional irregularities and disturbances arising during the process of loading. The problems of the loss of stability of elastic bars, plates, and shells in the presence of external perturbations are described by the equation (or system of equations)

$$A\Phi - \lambda B\Phi = \varphi \tag{1.1}$$

subject to the corresponding boundary conditions

$$C\Phi|_{\Gamma} = 0, \tag{1.2}$$

where Φ is the sought-for function; φ is a given function that characterizes small external perturbations; A and B are the quasilinear operators of the equations; C are the linear operators of the boundary conditions given on the contour Γ; λ is a load parameter. The function φ, which is defined on the middle surface of a thin-walled construction, describes the external perturbations upon loading. It should be emphasized that the boundary-value problem of the geometrically nonlinear theory of shells (1.1), (1.2) differs from the initial problem in that it has uniform boundary conditions.

Below we consider only the initial problems of buckling, whose boundary conditions are reduced to the form of (1.2) by selecting or constructing an appropriate (momentless-type) solution of the initial problem. The operators A and B have a rather complicated structure. Thus, after dedimensionalization the operator A contains a natural small parameter ε characterizing the thin-walled character of the shell. Note that the parameter ε appears as a multiplier in terms containing higher derivatives of the system of equations. The presence of the small parameter ε and the quasilinearity of the operators A and B may be expected to lead to a complicated structure of critical load spectra in linear problems of loss of stability in thin-walled constructions and to great variability in the postbuckling behavior of a nonlinear system after the loss of stability.

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The works of the London Symposium (1982) [2] on the theoretical and practical problems of the loss of stability in deformed members have revealed a great variety of approaches to the solution of theoretical problems and interpretation of the results obtained. In our opinion, this diversity of approaches and interpretations of results is associated with the above two circumstances. We will, therefore, introduce a classification of stability problems according to the structure of critical load spectra and according to the postbuckling behavior of a deformed system with finite deflections.

The critical loads $\lambda_i^{(0)}$ and the eigenforms $\Phi_i^{(0)}$ of the linear problems of loss of stability are the eigenvalues and eigenfunctions of the following problem:

$$A_0\Phi_i^{(0)} - \lambda_i^{(0)} B_0 \Phi_i^{(0)} = 0; \tag{1.3}$$

$$C\Phi_i^{(0)}|_\Gamma = 0. \tag{1.4}$$

Here A_0 and B_0 are linear differential operators with constant coefficients, the operator A_0 containing, for shells, a small parameter $\varepsilon \ll 1$ that is determined by the thin-walled character of the shell. The form of boundary conditions (1.4) coincides with that of boundary conditions (1.2). Let (1.3) and (1.4) be a self-adjoint problem. The eigenvalues (critical loads) are ordered in the standard way [3]:

$$0 < \lambda_1^{(0)} \leqslant \lambda_2^{(0)} \leqslant \lambda_3^{(0)} \leqslant \ldots \leqslant \lambda_i^{(0)} \leqslant \ldots, \tag{1.5}$$

and the eigenfunctions are orthogonal:

$$(\Phi_i^{(0)}, B_0\Phi_j^{(0)}) = \delta_{ij} \tag{1.6}$$

(δ_{ij} are the Kronecker symbols). In classifying the spectra of critical loads, the neighborhood of the smallest eigenvalue $\lambda_1^{(0)}$ corresponding to the Euler classical critical load is of the greatest practical interest. Therefore, we classify the spectra according to the multiplicity of the first eigenvalue and to the presence of a point of condensation in the initial part of the spectrum (when in the close neighborhood of $\lambda_1^{(0)}$ there are many other eigenvalues).

According to the presence of multiplicity there are two cases: 1) Ia, the first eigenvalue is separated from the second, i.e., $\lambda_1^{(0)}$ is a prime eigenvalue:

$$0 < \lambda_1^{(0)} < \lambda_2^{(0)} \leqslant \lambda_3^{(0)} \leqslant \ldots \leqslant \lambda_i^{(0)} \leqslant \ldots; \tag{1.7}$$

2) Ib, the first $j \geqslant 2$ eigenvalues coincide, i.e., $\lambda_1^{(0)}$ has multiplicity j:

$$0 < \lambda_1^{(0)} = \lambda_2^{(0)} = \ldots = \lambda_j^{(0)} < \lambda_{j+1}^{(0)} \leqslant \ldots . \tag{1.8}$$

According to the presence of a point of condensation in the initial part of the spectrum there can also be two cases: IIa, where there is no point of condensation in the initial part of the spectrum, and IIb where the spectrum starts with a point of condensation of critical loads [4, 5].

Recall that the cases Ia and IIa involve the problem of the loss of stability of a longitudinally compressed hinge-supported bar [1] (all the eigenvalues are different and the spectrum is sparse in the neighborhood of $\lambda_1^{(0)}$). In accordance with the suggested classification, the problem of the loss of stability of a longitudinally compressed hinge-supported bar lying on an elastic foundation can belong, depending on the geometric and stiffness parameters [1, 6], to one of the following types: Ia, IIa, or Ib, IIa, or Ia, IIb, or Ib, IIb. Note that the presence of the point of condensation precisely in this problem is most likely of no practical significance, since in this case the relationship of stiffness parameters is very exotic.

In problems of the stability of thin shells there are three types of critical load spectra [4]. Figures 1 and 2 present schematically the spectra of the above problems, respectively, in the absence of a condensation point in the neighborhood of $\lambda_1^{(0)}$ and in the presence of a condensation point at the start of the spectrum ($d\Lambda/d\lambda$ is the function characterizing the density of eigenvalues). Figure 1 shows the diagrams of loading cylindrical shells by hydrostatic and transverse pressure; the critical load spectra corresponding to these problems do not start from a condensation point.

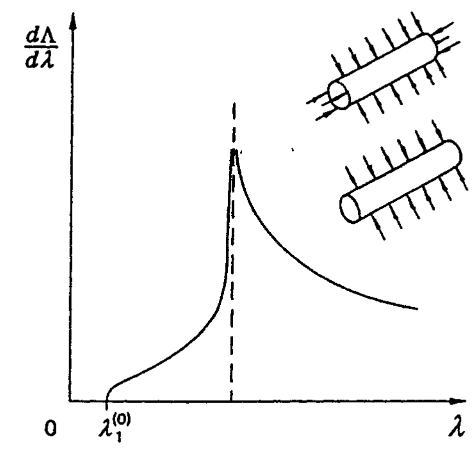

Fig. 1

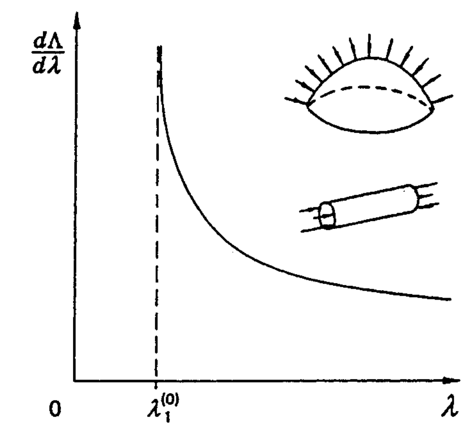

Fig. 2

Figure 2 depicts the diagrams of loading a spherical shell by hydrostatic pressure and a cylindrical shell by longitudinal compressive forces; the critical load spectra corresponding to these one-dimensional and two-dimensional problems start from a condensation point. Note the solvability conditions $(\varphi, B_0\Phi_i^{(0)}) = 0$ $(i = 1, 2, \ldots, j)$ of problem (1.1) and (1.2) when $A \equiv A_0$, $B \equiv B_0$, and $\lambda \to \lambda_1^{(0)}$; the number of these conditions j coincides with the multiplicity of the first eigenvalue [see (1.8)].

The above classification of the linear problems of buckling was tested by comparing the behavior of the simplest solutions [7] describing the stability loss process with the results of mass experiments on determining the critical load of constructively orthotropic, longitudinally compressed cylindrical shells [8]. The character of the spectrum in the neighborhood of $\lambda_1^{(0)}$ of the linear problems on the loss of stability of longitudinally compressed cylindrical orthotropic shells (presence or absence of condensation points) predicts very well the reproducibility and scatter of experimental results on the critical loads on thin-walled shells [7, 8]. In the case of finite deflections in a thin-walled system the initial region of the spectrum is distorted.

We now turn to the classification of the problems on the buckling of deformed constructions according to postbuckling behavior when a certain trivial solution branches into a prime solution ($\lambda_1^{(0)}$ is a prime eigenvalue) or complex solutions ($\lambda_1^{(0)}$ has multiplicity $j = 2$). The possible cases are as follows [1, 9]: IIIa, stable postbuckling behavior; IIIb, indifferent postbuckling behavior; IIIc, unstable postbuckling behavior (Fig. 3, respectively, curves a–c for an ideal system and curves a′–c′ for a system with perturbations). This reasoning is true only for small deviations of the system, i.e., at small norms $\|\Phi\|$ of the normal deflection of the system. The classification proposed is valid only for prime eigenvalues and eigenvalues divisible by 2; for $j > 2$ more precise definitions are necessary.

Thus, the entire range of problems on the buckling of deformed constructions is given by the diagram in Fig. 4. The stability of a longitudinally compressed hinge-supported bar Ia–IIIa is the simplest case: $\lambda_1^{(0)}$ is a prime eigenvalue; in the neighborhood of $\lambda_1^{(0)}$ the spectrum is sparse; the postbuckling behavior is stable at $\lambda > \lambda_1^{(0)}$. The most complicated stability problems appear to be the problem of the stability of a thin spherical orthotropic shell upon hydrostatic loading and the problem of the stability of a thin longitudinally compressed cylindrical orthotropic shell; as a rule, in the above classification these problems correspond to Ib, IIb, IIIc: $\lambda_1^{(0)}$ is a multiple eigenvalue; in the neighborhood of $\lambda_1^{(0)}$ there is a point of condensation; the postbuckling behavior is unstable at $\lambda > \lambda_1^{(0)}$. Recall that these problems have been solved in the classical statement without defining the modes of stability loss [10], the critical loads coinciding for one- and two-dimensional statements. Let us now pass on to the solution of problem (1.1), (1.2) for small finite deflections of the system being deformed. The quasilinear operators A and B, when there are small deflections, can be

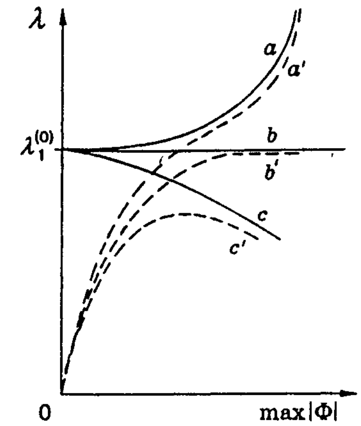

Fig. 3

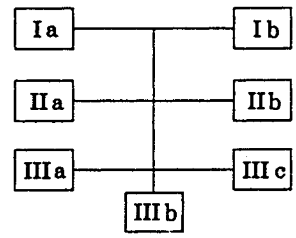

Fig. 4

written in the form

$$A = A_0 + \mu A_1 + \mu^2 A_2 + \dots, \quad B = B_0 + \mu B_1 + \mu^2 B_2 + \dots,$$

where A_0 and B_0 are linear operators [see Eq. (1.3)]; A_i and B_i ($i > 0$) are not necessarily linear differential operators; $\mu \ll 1$ is a small numerical parameter characterizing the amplitude of normal shell deflection. Many problems of the stability of mechanical systems can be reduced [1–3, 9] to the search for the eigenvalues and eigenfunctions of equations of the (1.1) type subject to certain boundary conditions. For instance, the equilibrium equation for the curved axis of a longitudinally compressed bar can be written [6, 11] in the form of (1.1) with a small parameter μ that characterizes the deviation of the system from the rectilinear (nonbuckled) state and is approximately proportional to the squared ratio of the deflection amplitude to the bar length. The linearized system of equations relating the function of stresses to the normal deflection of a nonideal shallow shell can also [12] be written in the form of (1.1) with a small parameter μ proportional to the amplitude of the initial irregularities.

At $\mu \not\equiv 0$ we obtain a problem, related to (1.3) and (1.4), on eigenvalues λ_i and functions Φ_i:

$$A\Phi_i - \lambda_i B\Phi_i = 0; \tag{1.9}$$

$$C\Phi_i|_\Gamma = 0. \tag{1.10}$$

The eigenvalues λ_i and the eigenfunctions Φ_i are determined for given finite deflections of the system, i.e., the small parameter μ and the operators A_j and B_j at $j \geqslant 1$ are assumed to be specified. The expressions for the eigenfunctions and the eigenvalues are sought for in the form of asymptotic series in terms of the parameter μ:

$$\Phi_n = \Phi_n^{(0)} + \sum_{k=1}^{\infty} \mu^k \Phi_n^{(k)}, \quad \lambda_n = \lambda_n^{(0)} + \sum_{k=1}^{\infty} \mu^k \lambda_n^{(k)}, \quad \Phi_n^{(k)} = \sum_{j=1}^{\infty} \beta_{nj}^{(k)} \Phi_j^{(0)}. \tag{1.11}$$

Equating the coefficients of the same powers of μ and taking into account the normalization conditions (1.6), one can formally [13, 14] find the coefficients of the expansions (1.11). Thus, in the neighborhood of $\lambda_n^{(0)}$ we obtain a parametric relationship of Φ_n and λ_n in terms of the small parameter μ. The results of testing of the above-described approach to the well-studied classical problem of calculating the deflection Φ of an ideal bar under longitudinal compressing load λ_1 in the neighborhood of the first critical load $\lambda_1^{(0)}$ [curve (a) in Fig. 3] are reported in [15]. A study of the buckling of an ideal bar on an elastic foundation using the perturbation technique [6, 11] has shown the possibility of unstable postbuckling behavior [curve (c) in Fig. 3], and experiments on real bars corroborated this possibility and demonstrated the reconstruction of the buckling modes, particularly when the corresponding eigenfunction-and-eigenvalue problem has multiple eigenvalues or a comparatively dense spectrum in the neighborhood of $\lambda_1^{(0)}$. In shell stability problems a simplified approach ignoring the spectrum density [5, 7] does not normally yield sufficient information on the behavior of a thin-walled construction upon buckling.

The solution of problem (1.1) and (1.2) is sought in the form of a series in terms of the specially constructed eigenfunctions Φ_i of problem (1.9) and (1.10) (see [3]):

$$\Phi = \sum_{i=1}^{\infty} a_i \Phi_i. \tag{1.12}$$

It appears expedient to retain in (1.12) all terms of the same order of smallness [5]. The buckling of real constructions in the presence of perturbations upon loading is accompanied by a gradual increase in the amplitude of the initial deflection [curves (a′–c′) in Fig. 3]. It is therefore suggested that the process of successive additional loading of a system being deformed should be combined with constructing the distorted region of the spectrum of the deformed system for finite deflection in each additional loading step. The additional loading step $\Delta\lambda$ is selected by the least eigenvalue $\lambda_1^{(0)}$ of the linear stability loss problem. In the first step of additional loading ($\lambda = \Delta\lambda$) in relationships (1.11) and (1.12) use is made of complete information on the eigenfunctions $\Phi_i^{(0)}$ and eigenvalues $\lambda_i^{(0)}$ of the classical stability problem (1.3) and (1.4); from the solution of (1.12) we find the small parameter $\mu^{(1)}$ (as a rule, it is connected with the amplitude of the normal deflection of the deformed system). In the second additional loading step ($\lambda = 2\Delta\lambda$), we first obtain exhaustive information on the initial part of the spectrum of problem (1.9), (1.10) for a nonideal system with the parameter $\mu^{(1)}$ and then, in (1.12), use the corrected eigenfunctions Φ_i and eigenvalues λ_i of problem (1.9), (1.10); from the constructed solution of (1.12) we find the small parameter $\mu^{(2)}$, etc. It is expedient to use the above approach only when there are analytical expressions for the solution of problem (1.3) and (1.4).

It is difficult to obtain any estimates for the accuracy of the suggested approximate method of constructing a solution in the general case. In a particular case [15], however, estimates of the accuracy can be obtained by comparing an exact solution of the problem with the suggested approximate solution. In [15] a detailed study is made of the problem on the buckling of an eccentrically compressed bar, i.e., of a nonideal construction with stable postbuckling behavior [curve (a′) in Fig. 3]. In the present work, the same combined approach is used to analyze the buckling of a construction with unstable postbuckling behavior [curve (c′) in Fig. 3].

2. Basic Equations. To study the process of buckling of shallow cylindrical shells, let us use the classical nonlinear system of equations with respect to the normal deflection and to the stress function [17]. We perform the standard nondimensionalization, i.e., divide the deflection by the shell thickness h, divide the lengths along the longitudinal coordinate x and along the circular coordinate y by the cylinder radius R, and nondimensionalize the stress function by means of the factor $(ERh^2)^{-1}$. After nondimensionalization the known normal deflection can be written as

$$w = \eta w^0(x, y), \quad \max|w^0(x, y)| = 1, \tag{2.1}$$

where η is a parameter that characterizes the deflection amplitude w.

Let relationship (2.1) be fulfilled in nonlinear terms of the transformed classical system of equations of shallow shell theory, but only for one of the factors. Then we have

$$\begin{gathered}\varepsilon^2 \Delta\Delta w + f_{xx} - \mu\left(f_{yy} w_{xx}^0 + f_{xx} w_{yy}^0 - 2 f_{xy} w_{xy}^0\right) - \lambda\left(a_1 w_{xx} + a_2 w_{yy}\right) = z, \\ \Delta\Delta f - w_{xx} + (1/2)\,\mu\left(w_{xx} w_{yy}^0 + w_{yy} w_{xx}^0 - 2 w_{xy} w_{xy}^0\right) = 0, \quad \Delta w = w_{xx} + w_{yy}, \\ \varepsilon^2 = [12(1-\nu^2)]^{-1}(h/R)^2, \quad \mu = \eta h/R, \quad 0 \leqslant x \leqslant L/R, \quad 0 \leqslant y \leqslant 2\pi\end{gathered} \tag{2.2}$$

under the corresponding boundary conditions (1.2). Here w and f are the normal deflection and the stress function; L is the length of the cylinder shell; λ is the loading parameter proportional to the constant component of the forces in the longitudinal and circular directions; a_1 and a_2 are the coefficients (for transverse loading $a_2 = -1$, $a_1 = 0$; for hydrostatic pressure $a_2 = -1$, $a_1 = -1/2$); z is the known additional load; E and

ν are the Young's modulus and the Poisson's coefficient; μ is a small parameter characterizing the imperfection of the system (at $\mu = 0$ the linear theory of shells is obtained). The nonlinear terms in the second equation of system (2.2) are written in symmetric form.

3. Description of the Technique of Constructing a Solution. System (2.2) can be written in the operator form

$$A\begin{bmatrix} w \\ f \end{bmatrix} - \lambda B\begin{bmatrix} w \\ f \end{bmatrix} = Z, \tag{3.1}$$

where the operators A and B, and the right-hand side Z are defined by the relations

$$A = A_0 + \mu A_1, \quad B = B_0,$$
$$A_0 = \begin{bmatrix} \varepsilon^2\Delta\Delta & (\ )_{xx} \\ -(\ )_{xx} & \Delta\Delta \end{bmatrix}, \quad B_0 = \begin{bmatrix} a_1(\ )_{xx} + a_2(\ )_{yy} & 0 \\ 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} z \\ 0 \end{bmatrix}, \tag{3.2}$$

$$A_1 = \begin{bmatrix} 0 & -[w^0_{xx}(\ )_{yy} + w^0_{yy}(\ )_{xx} - 2w^0_{xy}(\ )_{xy}] \\ (1/2)\,[w^0_{yy}(\ )_{xx} + w^0_{xx}(\ )_{yy} - 2w^0_{xy}(\ )_{xy}] & 0 \end{bmatrix}.$$

Note that the operator A consists of the main part and terms with the natural small parameter μ. For the deformation of a nonideal cylindrical shell, the function $w^0(x, y)$ in relationships (3.2) describes the initial deflection and, in the last row of the operator A_1, the coefficient 1/2 is replaced by 1.

We assume that all the coefficients of the operator A_1 are known, i.e., we seek a solution for Eq. (3.1) on a given function w^0 having continuous second derivatives. Then it is natural to construct the solution in the form of the series

$$\begin{bmatrix} w \\ f \end{bmatrix} = \sum_{i=1}^{\infty} a_i \begin{bmatrix} w_i \\ f_i \end{bmatrix} \tag{3.3}$$

in terms of stability loss eigenforms if the latter are known or can readily be constructed. Substituting the expression of solution (3.3) into Eq. (3.1), after obvious transformations, we obtain a relationship for determining the coefficients a_i of the Fourier series:

$$a_i = z_i/(\lambda_i - \lambda). \tag{3.4}$$

Here λ_i are the eigenvalues of the stability loss problem

$$A\begin{bmatrix} w_i \\ f_i \end{bmatrix} - \lambda_i B\begin{bmatrix} w_i \\ f_i \end{bmatrix} = 0, \quad C\begin{bmatrix} w_i \\ f_i \end{bmatrix}\Bigg|_{\Gamma} = 0, \tag{3.5}$$

to which correspond the eigenfunctions w_i and f_i; A and B are the operators defined in (3.2); and C is the operator of the boundary conditions corresponding to the conditions of the hinge support of the ends and to the momentless prebuckling state; the constants z_i are determined from the simple relationships

$$z_i = \left(\begin{bmatrix} z \\ 0 \end{bmatrix}, \begin{bmatrix} w_i \\ f_i \end{bmatrix}\right). \tag{3.6}$$

It is assumed that for the eigenfunctions the following conditions of orthogonality and normalization are fulfilled:

$$\left(B\begin{bmatrix} w_i \\ f_i \end{bmatrix}, \begin{bmatrix} w_j \\ f_j \end{bmatrix}\right) = \delta_{ij}, \quad \left(A\begin{bmatrix} w_i \\ f_i \end{bmatrix}, \begin{bmatrix} w_j \\ f_j \end{bmatrix}\right) = \lambda_i\delta_{ij}. \tag{3.7}$$

In relationships (3.6) and (3.7) by the scalar product is meant the functional

$$\left(\begin{bmatrix} w_i \\ f_i \end{bmatrix}, \begin{bmatrix} w_j \\ f_j \end{bmatrix}\right) = \int_0^{2\pi}\int_0^{L/R} (w_i w_j + f_i f_j)\, dx dy.$$

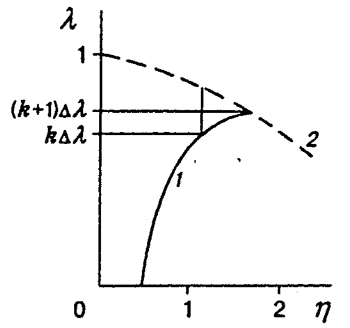

Fig. 5

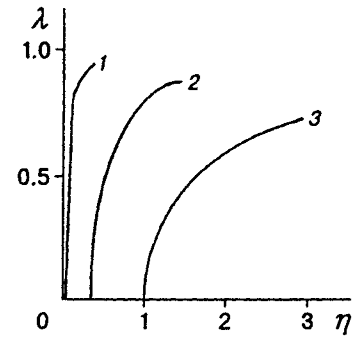

Fig. 6

If solution (3.3) has already been constructed for an arbitrary loading parameter λ and the critical loading parameter λ^* at which the system loses stability has been determined, this solution can readily be analyzed for a particular λ, because the decrease in the coefficients a_i of the Fourier series is determined by the assigned transverse load and by the spectrum of problem (3.5) (see [4, 7, 18]).

4. Study of the Initial Region of the Critical Load Spectrum. Let us consider the numerical construction of the spectrum. The operators A and B of problem (3.5) contains terms with the small parameter μ, and hence it is natural to use the classical perturbation theory [13, 14] and to write the eigenfunctions w_i and f_i and eigenvalues λ_i in the form of asymptotic series in terms of the parameter μ:

$$w_i = w_i^{(0)} + \sum_{k=1}^{\infty} \mu^k w_i^{(k)}, \quad f_i = f_i^{(0)} + \sum_{k=1}^{\infty} \mu^k f_i^{(k)}, \quad \lambda_i = \lambda_i^{(0)} + \sum_{k=1}^{\infty} \mu^k \lambda_i^{(k)}, \quad \mu \ll 1. \tag{4.1}$$

We restrict relationships (4.1) to first-order infinitesimal terms.

To effectively construct the initial region of the spectrum by the perturbation method it is desirable to have analytical expressions for the eigenfunctions $w_i^{(0)}$ and $f_i^{(0)}$ and eigenvalues $\lambda_i^{(0)}$ of the unperturbed problem (1.3) and (1.4), whose eigenvalues are at least multiples of 2:

$$\lambda_1^{(0)} = \lambda_2^{(0)} \leqslant \lambda_3^{(0)} = \lambda_4(0) \leqslant \ldots \leqslant \lambda_i^{(0)} = \lambda_{i+1}^{(0)} \leqslant \ldots, \tag{4.2}$$

because, e.g., the component

$$w_i^{(0)} = \begin{cases} \gamma_{n_i} \cos n_i y \cdot \sin \pi Rx/L, \\ \gamma_{n_i} \sin n_i y \cdot \sin \pi Rx/L, \end{cases} \quad \gamma_{n_i} = \frac{1}{n_i}\sqrt{\frac{2R}{\pi L}}. \tag{4.3}$$

In this case, to the odd i in (4.2) and (4.3) corresponds $\cos n_i y$ and to the even i, $\sin n_i y$ (n_i is the number of waves along the circumferential coordinate). According to the classification in Section 1, the problems of the buckling of a cylindrical shell of mean length belong to the class Ib, IIa, IIIc, and hence in their numerical realization a limited number of degrees of freedom and the instability of the postbuckling behavior of the system are taken into account. In the presence of perturbations, the multiple eigenvalues (4.2) are prime if in expansions (4.1) we retain all terms up to $\cos 2n_i y$ and $\sin 2n_i y$. The corrections to the eigenvalues have different signs. Hence, the least critical load for a nonideal system is less than for an ideal system and, therefore, under successive additional loading the real system loses its stability sooner than the ideal one.

The results of calculation of the initial region of the spectrum using perturbation theory agree well with the result of analytical investigation of this region by the Courant method [4, 7, 19].

5. Numerical Construction of the Solution and Discussion. When constructing a numerical solution, it is necessary to curtail series (3.3) so that the main information could be retained. To this end, the simplest plan to be followed is to compare the absolute values of the coefficients a_i of the Fourier series. We

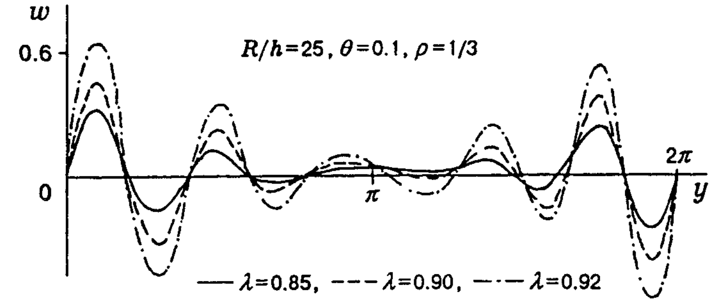

Fig. 7

retain all terms of the expansion for which

$$|a_j| \geqslant \rho \max_i |a_i|. \tag{5.1}$$

In the calculations performed the constant ρ was chosen to be equal to 1/2, 1/3, and 1/10. Retaining a finite number of terms of series (3.3), we replace the system with distributed parameters (with an infinite number of degrees of freedom) by a system with a finite number of degrees of freedom, all of them equal in status.

Numerical analysis of the process of buckling is performed as follows. First, a step for the load λ is chosen (e.g., $\Delta\lambda = 0.01 \min \lambda_i^{(0)}$). In each kth step, the solution of (3.3) is constructed. We determine the parameter η [see (2.1)], calculate the small parameter $\mu = \eta h/R$, and refine the eigenforms of stability loss and the critical loads of a nonideal system using the perturbation method at normal deflections known from the $(k-1)$th step.

We further pass over to the $(k+1)$th step for a load equal to $(k+1)\Delta\lambda$, the function w^0 for this step being taken from the kth step. Calculations start from the first step ($k = 1$), while $w^0 \equiv 0$ for the zeroth step. The numerical calculations end when the load parameter $(k+1)\Delta\lambda$ in the next $(k+1)$th step exceeds or coincides with the least critical load λ_1 determined in the previous kth step for a nonideal system, i.e., $(k+1)\Delta\lambda \geqslant \lambda_1$; this buckling mode corresponds to the unstable postbuckling behavior of the system. The critical load is taken to be $\lambda^* = k\Delta\lambda$ (Fig. 5, curve 1 describes the buckling of the system and curve 2, the changes in λ_1).

Presented below are the calculation results for shells loaded by a uniform external transverse pressure and a comparatively low additional external normal pressure:

$$z = \theta \sum_{n=4}^{100} a_n (\cos ny + \sin ny) \sin \frac{\pi R x}{L}, \quad a_n = \frac{2R}{n^2 \pi L \gamma_n}, \quad \gamma_n = \frac{1}{n}\sqrt{\frac{2R}{\pi L}}. \tag{5.2}$$

Since the classical nonlinear equations are the subject of investigation, all states with a small index of variability ($n < 4$) along the circular coordinate are excluded from consideration. Note that in every kth step ($k > 1$) the function z defined in (5.2) is reexpanded as a series of refined eigenforms of stability loss.

Figure 6 presents typical deformation curves 1–3 ($\theta = 0.01, 0.1, 0.3$) at $R/h = 100$, $\rho = 1/3$, $L/R = 1$. Tables 1–3 list the results of determination of the critical load λ^*. Here n^* is the number of waves along the circular coordinate for the least critical load of the linear problem; n_j are the mode numbers satisfying relationship (5.1) in the last step of additional loading; symbols s and c stand for sin and cos; and the underlined mode number corresponds to $\max |a_i|$. The calculations for Tables 1 and 2 were performed at $\rho = 1/3$ and $L/R = 1$, and for Tables 2 and 3, at $R/h = 100$ ($n^* = 8$). The buckling modes are found to be reconstructed upon additional loading. This reconstruction is associated with a deviation of the spectrum of the nonlinear system upon finite deflections compared with the initial system. All the calculation results of Table 1 were duplicated at $\rho = 1/2$ and 1/10 ($L/R = 1$). The results that, in our opinion, are the most

TABLE 1

θ	R/h					
	25 ($n^* = 6$)		50 ($n^* = 7$)		100 ($n^* = 8$)	
	λ^*	n_j	λ^*	n_j	λ^*	n_j
0.01	0.97	5c, 5s, 6c, $\underline{6}$s	0.97	7c, $\underline{7}$s	0.96	8c, $\underline{8}$s
0.1	0.92	$\underline{5}$s, 6c, 6s	0.90	6s, 7c, $\underline{7}$s	0,86	7s, $\underline{8}$s, 9s
1	0.62	4s, $\underline{5}$s, 6s	0.53	5s, $\underline{6}$s, 7s	0.42	$\underline{7}$s, 8s

TABLE 2

θ	η	λ^*	n_j
0.1	1.56	0.86	7s, $\underline{8}$s, 9s
0.3	3.56	0.75	7s, $\underline{8}$s, 9s
0.5	5.17	0.65	$\underline{7}$s, 8s
0.6	5.70	0.60	$\underline{7}$s, 8s
0.7	5.80	0.55	$\underline{7}$s, 8s
0.8	5.82	0.50	6s, $\underline{7}$s, 8s
0.9	6.31	0.46	6s, $\underline{7}$s, 8s
1.0	6.68	0.42	$\underline{7}$s, 8s

TABLE 3

ρ	λ^*	n_j
1/2	0.43	$\underline{7}$s
1/3	0.42	$\underline{7}$s, 8s
1/10	0.42	4s, ..., $\underline{7}$s, ..., 11s

interesting are listed in Table 3; they show that it often suffices to restrict oneself to the parameter $\rho \leqslant 1/3$ in determining the number of equivalent degrees of freedom [see relationship (5.1)]. The calculation results that are not given in Table 3 completely coincide in the magnitude of the critical load. Since perturbation theory was used in determining λ, it is desirable to refine the calculations in the cases where $\lambda < 0.6$.

In calculating and analyzing the buckling problems we took into account the following: 1) the multiplicity of the first eigenvalue, 2) the spectrum density in the neighborhood of the first eigenvalue, 3) the postbuckling behavior of the system (see Section 1).

Figure 7 presents the typical deflections along the circumferential coordinate for the specified parameters, which corresponds to Table 1; one can readily notice the reconstruction of the buckling modes upon successive additional loading from 0.85 to 0.92 of the Euler load.

The above analysis of the buckling of cylindrical shells shows the process of additional loading to be accompanied by distortion of the initial region of the critical load spectrum and by reconstruction of the buckling modes. The distortion of the spectrum is due to both additional forces arising in the middle surface and additional normal deflections. The reconstruction of the buckling modes is due to the distortion of the spectrum of the nonlinear system upon finite deflections in comparison with the spectrum of the initial system.

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